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# A CONVEX APPROXIMATION FOR TWO-STAGE MIXED-INTEGER RECURSE MODELS WITH A UNIFORM ERROR BOUND\*

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**Abstract.** We develop a convex approximation for two-stage mixed-integer recourse models, and we derive an error bound for this approximation that depends on the total variations of the probability density functions of the random variables in the model. We show that the error bound converges to zero if all these total variations converge to zero. Our convex approximation is a generalization of the one in Romeijnders, van der Vlerk, and Klein Haneveld [*Math. Program.*, to appear] restricted to totally unimodular integer recourse models. For this special case it has the best worst-case error bound possible. The error bound in this paper is the first in the general setting of mixed-integer recourse models. As main building blocks in its derivation we generalize the asymptotic periodicity results of Gomory [*Linear Algebra Appl.*, 2 (1969), pp. 451–558] for pure integer programs to the mixed-integer case, and we use the total variation error bounds on the expectation of periodic functions derived in Romeijnders, van der Vlerk, and Klein Haneveld [*Math. Program.*, to appear].

**Key words.** stochastic programming, mixed-integer recourse, convex approximation

**AMS subject classifications.** 90C15, 90C10

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**1. Introduction.** We consider the two-stage mixed-integer recourse model (with random right-hand side only):

$$\min\{cx + Q(z) : Ax = b, z = Tx, x \in \mathbb{R}_+^{n_1}\},$$

where, for tender variables  $z \in \mathbb{R}^m$ ,

$$(1.1) \quad Q(z) := \mathbb{E}_\omega[v(\omega - z)], \quad z \in \mathbb{R}^m,$$

and

$$(1.2) \quad v(s) := \min\{qy : Wy = s, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}\}, \quad s \in \mathbb{R}^m.$$

The functions  $Q$  and  $v$  are called the recourse function and second-stage value function, respectively. They represent the (expected) costs of the so-called recourse actions  $y$  for compensating infeasibilities of the random goal constraints  $Tx = \omega$ . We assume throughout that  $W$  is an integer matrix and that  $\omega$  is a continuous random vector with joint probability density function (pdf)  $f$ . Moreover, we focus on mixed-integer recourse models having integer restrictions on (some of) the recourse actions  $y$ , and for ease of exposition we disregard any integer decision variables in the first stage; the results presented in this paper also hold without this latter assumption.

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Many practical problems can be cast within this framework; see, e.g., [6, 9, 26] for examples in logistics, energy, and finance. In these problems integer decision variables may arise naturally to model indivisibilities or on/off decisions. However, solving such recourse problems with integer decision variables poses additional challenges over continuous recourse problems, since the latter are convex (and thus the rich toolbox of convex optimization can be used) whereas the former are generally not [15].

To overcome this difficulty we use convex approximations  $\hat{Q}$  for the recourse function  $Q$ . The idea is to construct an approximating model that (1) is convex, so that the model can be solved efficiently, and (2) is a close approximation of the original mixed-integer recourse model, so that we obtain good or even (near-)optimal first-stage decisions  $x$ . We will contribute to this second objective by deriving an error bound for  $\hat{Q}$ . To be precise, we derive an upper bound for  $\|Q - \hat{Q}\|_\infty := \sup_{z \in \mathbb{R}^m} |Q(z) - \hat{Q}(z)|$ .

Convex approximations of mixed-integer recourse models were first developed for the special case of simple integer recourse models [10] where the recourse matrix  $W$  is separable, and were later extended to the totally unimodular integer [23] and simple mixed-integer [24] cases. The key idea in these approximations is to simultaneously relax the integrality constraints in the second-stage model and perturb the distribution of the random right-hand side  $\omega$  to obtain a convex recourse function  $\hat{Q}$  corresponding to a continuous recourse model, for which efficient algorithms are available.

Recently, significant progress has been made in deriving error bounds for these approximations in the case when the recourse matrix  $W$  has a nonseparable structure. For example, Romeijnnders, van der Vlerk, and Klein Haneveld [18] derive an error bound for the convex approximations of [23] for the totally unimodular (TU) integer case that depends on all total variations of the density functions of the random variables in the model. The main building block in the derivation is a total variation error bound on the expectation of a particular one-dimensional two-valued periodic function. In [17] this result is generalized to arbitrary one-dimensional periodic functions leading to, among others, a new convex approximation for TU integer recourse models with an error bound that is tight in a worst-case sense.

In this paper we construct a convex approximation  $\hat{Q}$  for general two-stage mixed-integer recourse models with random right-hand side. We derive an error bound for  $\hat{Q}$  that converges to zero as all total variations of the probability density functions of the random variables in the model converge to zero. To derive this error bound we use asymptotic periodicity of the underlying mixed-integer value function (generalizing results in Gomory [7] for the pure integer case) and the total variation bounds on the expectation of periodic functions of [17].

Our approach differs considerably from alternative approaches in the literature for solving mixed-integer recourse models. These alternative approaches typically combine solution methods from deterministic integer programming and stochastic continuous programming to obtain (near-)optimal solutions, and have difficulty solving large problem instances. Examples of such methods are dual decomposition [3] and the integer  $L$ -shaped method [12]. We do not elaborate on these and other methods here. The interested reader is referred to [1, 5, 8, 14, 20, 22] or the survey papers Klein Haneveld and van der Vlerk [11], Louveaux and Schultz [13], Romeijnnders, Stougie, and van der Vlerk [16], Schultz [19], and Sen [21].

In the remainder of this paper we first discuss asymptotic periodicity of the mixed-integer value function  $v$  in section 2. We derive this result by studying so-called Gomory relaxations of  $v$ , and by showing, among other results, that the optimal

solutions of these relaxations are periodic and bounded. Next, we use these results in section 3 to derive a convex approximation  $\hat{v}$  of  $v$ , and consequently, a convex approximation  $\hat{Q}$  of  $Q$ , such that the difference  $v - \hat{v}$  is periodic on particular convex subsets of its domain. This allows us to use the total variation error bounds on the expectation of periodic functions developed in [17] to derive an upper bound on  $\|Q - \hat{Q}\|_\infty$  in section 5. In sections 3 and 4 we present auxiliary results to derive this bound. Those in section 3 concern the approximating value function  $\hat{v}$  and those in section 4 the total variation error bounds. We end this paper with a discussion in section 6.

Throughout this paper we make the following assumptions.

- (A1) The recourse is complete, i.e., for every  $s \in \mathbb{R}^m$  there exists a feasible recourse action  $y$ .
  - (A2) Dual feasibility of the LP-relaxation:  $\{\lambda \in \mathbb{R}^m : \lambda W \leq q\} \neq \emptyset$ .
  - (A3) Finite first moment:  $\int_{\mathbb{R}^m} \|u\|_2 f(u) du < +\infty$ .
- Assumptions (A1) and (A2) ensure that  $v(s)$  is finite for all  $s \in \mathbb{R}^m$ , and (A1)–(A3) imply that  $Q(z)$  is finite for all  $z \in \mathbb{R}^m$ .

**2. Asymptotic periodicity in mixed-integer linear programming.** In this section we derive an asymptotic periodicity result for the value function of a mixed-integer linear programming problem. The result, given in Theorem 2.9, is similar to the results of Gomory [7] for the pure integer case and of Wolsey [27] for the mixed-integer case with integer right-hand side only. To the best of our knowledge the generalization to mixed-integer linear programming problems presented here is new. Although the main line of the proof is the same as in Gomory [7], some adjustments have been made to deal with this more general case. Throughout, we will point out these differences.

We consider the optimal value function  $v$  of a mixed-integer linear program,

$$v(s) = \min\{qy : Wy = s, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}\}, \quad s \in \mathbb{R}^m,$$

as defined in (1.2). In our terminology we do not distinguish between a value function and its associated optimization problem. For example, we call the value function  $v_{LP}$  defined as

$$v_{LP}(s) := \min\{qy : Wy = s, y \in \mathbb{R}_+^{n_2} \times \mathbb{R}_+^{n_3}\}, \quad s \in \mathbb{R}^m,$$

the LP-relaxation of  $v$ .

As already mentioned, we assume that  $W$  is an integer matrix. Moreover, since  $v(s)$  is finite for all  $s \in \mathbb{R}^m$  by (A1) and (A2), we have  $n_2 + n_3 \geq m + 1$ , and we define  $n := n_2 + n_3 - m$ , so  $n \geq 1$ .

In the remainder of this section we define the Gomory relaxation  $v_B$  of  $v$  in section 2.1, and we derive properties, such as periodicity and boundedness, of the optimal solutions of this relaxation in section 2.2. Moreover, we derive a sufficient condition under which the optimal solutions of the Gomory relaxation are also optimal for the mixed-integer value function  $v$ . Using these results, we derive the asymptotic periodicity of the mixed-integer value function  $v$  in section 2.3.

**2.1. The Gomory relaxation  $v_B$ .** Let  $B$  be a dual feasible basis matrix of  $v_{LP}$ , the LP-relaxation of  $v$ . Then,  $W \equiv [B \ N]$ , meaning that the left- and right-hand sides are equal up to a permutation of the columns. Using the same permutation, we have  $q \equiv [q_B \ q_N]$  and  $y \equiv [y_B \ y_N]$ , and we assume that the permutation is such that only the first  $n_B$  and  $n_N$  components of  $y_B$  and  $y_N$ , respectively, are restricted to

being integers. Then, we can rewrite the mixed-integer value function  $v(s)$  for every  $s \in \mathbb{R}^m$  as

$$v(s) = \min \left\{ q_B y_B + q_N y_N : B y_B + N y_N = s, y_B \in \mathbb{Z}_+^{n_B} \times \mathbb{R}_+^{m-n_B}, y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N} \right\}.$$

Since  $B$  is a basis matrix it is nonsingular, and we can substitute  $y_B = B^{-1}(s - N y_N)$  to obtain for every  $s \in \mathbb{R}^m$ ,

$$\begin{aligned} v(s) = \min_{y_N} \quad & \bar{q}_N y_N + q_B B^{-1} s \\ \text{s.t.} \quad & B^{-1} s - B^{-1} N y_N \in \mathbb{Z}_+^{n_B} \times \mathbb{R}_+^{m-n_B}, \\ & y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}, \end{aligned}$$

where  $\bar{q}_N := q_N - q_B B^{-1} N$  denote the reduced costs of  $y_N$ . Since  $B$  is a dual feasible basis matrix, we have  $\bar{q}_N \geq 0$ . This implies that the solution  $y_N = 0$  is optimal for  $v_{LP}(s)$  if  $B^{-1} s \geq 0$ , with objective value  $q_B B^{-1} s$ . In the mixed-integer case the nonbasic variables  $y_N$  are typically not all zero in any optimal solution. We use the relaxation introduced by Gomory [7] to derive properties of these nonbasic solutions, parametrically in  $s$ . This *Gomory relaxation* is obtained by relaxing the nonnegativity constraints on  $y_B$ .

**DEFINITION 2.1.** Consider the mixed-integer value function  $v$  defined in (1.2) and let  $B$  denote a dual feasible basis matrix of its LP-relaxation  $v_{LP}$ . For any  $s \in \mathbb{R}^m$ , we define the Gomory relaxation of  $v(s)$  with respect to  $B$  as  $v_B(s)$ , given by

$$\begin{aligned} v_B(s) := \min_{y_N} \quad & \bar{q}_N y_N + q_B B^{-1} s \\ \text{s.t.} \quad & B^{-1} s - B^{-1} N y_N \in \mathbb{Z}_+^{n_B} \times \mathbb{R}_+^{m-n_B}, \\ & y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}, \end{aligned}$$

where  $\bar{q}_N := q_N - q_B B^{-1} N \geq 0$  denote the reduced costs of  $y_N$ .

Notice that every optimal solution  $y_N^*(s)$  of the Gomory relaxation  $v_B(s)$  is optimal for the mixed-integer value function  $v(s)$  if  $B^{-1} s - B^{-1} N y_N^*(s) \geq 0$ . Thus, by deriving properties of the optimal solutions  $y_N^*(s)$  of  $v_B(s)$  we also obtain properties of  $v(s)$  for those  $s \in \mathbb{R}^m$  satisfying  $B^{-1} s - B^{-1} N y_N^*(s) \geq 0$ . Below we will derive these properties.

**2.2. Properties of the Gomory relaxation  $v_B$ .** At this point we will deviate from the work of Gomory [7] and Wolsey [27]. For the pure integer case Gomory introduces a group equation to model the constraints of  $v_B$ . He shows that the optimal solutions  $y_N^*(s)$  are uniformly bounded and periodic in  $s$ . Wolsey [27] obtains similar results for the mixed-integer case with integer right-hand side by deriving an equivalent pure integer programming problem for the mixed-integer value function, and applying group theory to this pure integer program. To deal with the more general mixed-integer value function we use an alternative line of reasoning. We will show that also for general mixed-integer value functions the optimal solutions  $y_N^*(s)$  of the Gomory relaxation  $v_B(s)$  are periodic and bounded uniformly in  $s$ . It is not surprising that these properties arise:  $y_N^*(s)$  is periodic since the optimization problem in  $v_B(s)$  is identical for  $s = s_1$  and  $s = s_2$  if the fractional values of the vectors  $B^{-1} s_1$  and  $B^{-1} s_2$  are equal, and  $y_N^*(s)$  is uniformly bounded since the cost coefficients  $\bar{q}_N \geq 0$ , the variables  $y_N \geq 0$ , and the objective function is to be minimized.

Before we prove these properties in Lemma 2.3, we first give a definition of  $B$ -periodicity in the spirit of Gomory [7]. Moreover, we let  $\det(B)$  denote the determinant of  $B$  and  $\text{adj}(B)$  its adjoint.

DEFINITION 2.2. *Let the function  $g : \mathbb{R}^m \mapsto \mathbb{R}^n$  be given and let  $B$  be an  $m \times m$  matrix. Then,  $g$  is called  $B$ -periodic if and only if for every  $x \in \mathbb{R}^m$  and  $\ell \in \mathbb{Z}^m$*

$$g(x) = g(x + B\ell).$$

LEMMA 2.3. *Let  $B$  be a dual feasible basis matrix of the LP-relaxation of  $v$  and consider its Gomory relaxation  $v_B$ . Assume that (A1) and (A2) hold. Then, there exists a function  $y_N^*(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^n$  such that*

- (i)  $y_N^*(s)$  is optimal for  $v_B(s)$  for every  $s \in \mathbb{R}^m$ ,
- (ii)  $y_N^*(s) \in [0, p]^n$  for every  $s \in \mathbb{R}^m$  with  $p = |\det(B)|$ , and
- (iii)  $y_N^*(\cdot)$  is  $B$ -periodic.

*Proof.* Let  $s \in \mathbb{R}^m$  be given and let  $y_N(s)$  be a feasible solution of  $v_B(s)$ . Such a feasible solution exists by assumption (A1). We will show that  $y'_N(s) := y_N(s) - p \lfloor p^{-1} y_N(s) \rfloor$  is also feasible and has an objective value at least as good as  $y_N(s)$ . Moreover, since (componentwise)  $y'_N(s) = y_N(s)$  modulo  $p$ , we have  $y'_N(s) \in [0, p]^n$ .

Feasibility of  $y'_N(s)$  follows from the observations that  $p \in \mathbb{Z}$ , since  $p = |\det(B)|$  and  $B$  is an integer matrix, and  $B^{-1}s - B^{-1}Ny'_N(s) \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B}$ . This latter observation is true since, using  $B^{-1} = (\det(B))^{-1}\text{adj}(B)$ ,

$$B^{-1}s - B^{-1}Ny'_N(s) = (B^{-1}s - B^{-1}Ny_N(s)) + \frac{|\det(B)|}{\det(B)} \text{adj}(B)N \lfloor p^{-1} y_N(s) \rfloor,$$

and both  $\text{adj}(B)$  and  $N$  are integer matrices.

The improvement in objective value follows since  $y'_N(s) \leq y_N(s)$  and  $\bar{q}_N \geq 0$ . We conclude that without loss of generality we can rewrite the Gomory relaxation  $v_B$  as

$$(2.1) \quad \begin{aligned} v_B(s) &:= \min_{y_N} \quad \bar{q}_N y_N + q_B B^{-1}s \\ \text{s.t.} \quad & B^{-1}s - B^{-1}Ny_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B}, \\ & y_N \in [0, p]^n, \\ & y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}. \end{aligned}$$

This optimization problem can be considered as minimizing a continuous function over a compact set, and thus it follows from Weierstrass' theorem that (i) an optimal solution  $y_N^*(s)$  of  $v_B(s)$  exists for every  $s \in \mathbb{R}$ . This optimal solution satisfies (ii)  $y_N^*(s) \in [0, p]^n$ . Moreover, we can choose  $y_N^*(\cdot)$  such that (iii)  $y_N^*(\cdot)$  is  $B$ -periodic since the fractional values of  $B^{-1}(s + B\ell)$  and  $B^{-1}s$  are equal for every  $\ell \in \mathbb{Z}^m$ , and thus the optimization problems  $v_B(s + B\ell)$  and  $v_B(s)$  are the same up to a constant for every  $\ell \in \mathbb{Z}^m$ .  $\square$

Remark 2.4. We realize that the function  $y_N^*(\cdot)$  is not necessarily unique since the Gomory relaxation  $v_B(s)$  may have multiple optimal solutions. Nonetheless, we will refer to  $y_N^*(s)$  as the optimal solution of  $v_B(s)$ , with the understanding that  $y_N^*(\cdot)$  satisfies properties (i)–(iii) of Lemma 2.3.

Using the properties of  $y_N^*(\cdot)$  in Lemma 2.3, it is not hard to derive a sufficient condition on  $s$  so that  $y_N^*(s)$  is not only optimal for  $v_B(s)$  but also for  $v(s)$ . This sufficient condition will guarantee that  $B^{-1}s - B^{-1}Ny_N^*(s) \geq 0$  holds. Similarly to Gomory [7], we will make use of the fact that  $\|y_N^*(s)\|_\infty$  is bounded uniformly in  $s$  by Lemma 2.3 (ii). We define  $\Lambda := \{t \in \mathbb{R}^m : B^{-1}t \geq 0\}$ , so that  $y_N^*(s)$  is optimal for



$v(s)$  if  $s - Ny_N^*(s) \in \Lambda$ . Clearly, if the distance from  $s \in \Lambda$  to the boundary of  $\Lambda$  is large enough, then  $s - Ny_N^*(s) \in \Lambda$ , motivating the following definition.

**DEFINITION 2.5.** Let  $\Lambda \subset \mathbb{R}^m$  be a closed convex cone and let  $d \in \mathbb{R}$  with  $d > 0$  be given. Then, we define  $\Lambda(d)$  as

$$\Lambda(d) := \{s \in \Lambda : \mathcal{B}(s, d) \subset \Lambda\},$$

where  $\mathcal{B}(s, d) := \{t \in \mathbb{R}^m : \|t - s\|_2 \leq d\}$  is the closed ball centered at  $s$  with radius  $d$ . We can interpret  $\Lambda(d)$  as the set of points in  $\Lambda$  with at least Euclidean distance  $d$  to the boundary of  $\Lambda$ .

**Example 2.6.** Let  $H = \{x \in \mathbb{R}^m : a^T x \geq 0\}$  be a closed halfspace through the origin with normal vector  $a \neq 0$ . Then, it follows from elementary geometry that for  $d \in \mathbb{R}$  with  $d > 0$ ,  $H(d) = \{x \in \mathbb{R}^m : a^T x \geq d\|a\|_2\}$ . That is,  $H(d)$  is a closed halfspace with the same normal vector  $a$  as  $H$ , but the boundary of  $H(d)$  is shifted by a distance  $d$  in the direction of the normal vector  $a$ .

**Example 2.7.** Let  $\Lambda = \{x \in \mathbb{R}^m : Ax \geq 0\}$  be a closed convex polyhedral cone with  $A \in \mathbb{R}^{m \times m}$  nonsingular. Then,  $\Lambda$  is the intersection of  $m$  closed halfspaces  $H_i = \{x \in \mathbb{R}^m : a_i^T x \geq 0\}$ ,  $i = 1, \dots, m$ , where  $a_i^T$  denotes the  $i$ th row of  $A$ . Since  $s \in \Lambda(d)$  if and only if  $s \in H_i(d)$  for all  $i = 1, \dots, m$ , it follows from Example 2.6 that  $\Lambda(d) = \{x \in \mathbb{R}^m : Ax \geq b\}$  with  $b_i = d\|a_i\|_2$ . Thus,  $\Lambda(d)$  is a closed convex set with the same shape as  $\Lambda$ , but shifted by a vector  $A^{-1}b$ , i.e.,  $\Lambda(d) = A^{-1}b + \Lambda$ .

**LEMMA 2.8.** Let  $B$  denote a dual feasible basis matrix of the LP-relaxation of the mixed-integer value function  $v$  defined in (1.2). For every  $s \in \mathbb{R}^m$ , let  $y_N^*(s)$  denote the optimal solution to the Gomory relaxation  $v_B(s)$ , with  $y_N^*(s)$  satisfying the properties in Lemma 2.3. Then, for the closed convex polyhedral cone  $\Lambda := \{t \in \mathbb{R}^m : B^{-1}t \geq 0\}$  and distance  $\bar{d} := |\det(B)| \sum_{j=1}^n \|N_j\|_2$  with  $N_j$  denoting the  $j$ th column of  $N$ , we have for every  $s \in \Lambda(\bar{d})$  that

- (i)  $s - Ny_N^*(s) \in \Lambda$ , and
- (ii)  $y_N^*(s)$  is an optimal solution of  $v(s)$ .

*Proof.* Let  $s \in \Lambda(\bar{d})$  be given and define  $s' := s - Ny_N^*(s)$ . We will prove (i) by showing that  $\|s' - s\|_2 \leq \bar{d}$  and thus, by the definition of  $\Lambda(\bar{d})$ , we have  $s' \in \mathcal{B}(s, \bar{d}) \subset \Lambda$ . Rewriting  $\|s' - s\|_2$  yields

$$\|s' - s\|_2 = \|Ny_N^*(s)\|_2 = \left\| \sum_{j=1}^n y_j^*(s) N_j \right\|_2,$$

where  $y_j^*(s)$  denotes the  $j$ th component of  $y_N^*(s)$ . Applying the triangle inequality and using  $|y_j^*(s)| \leq |\det(B)|$  by Lemma 2.3 (ii), we obtain

$$\|s' - s\|_2 \leq \sum_{j=1}^n |y_j^*(s)| \|N_j\|_2 \leq |\det(B)| \sum_{j=1}^n \|N_j\|_2 = \bar{d}.$$

Hence, (i)  $s - Ny_N^*(s) \in \Lambda$ , and by definition of  $\Lambda$  we have  $B^{-1}s - B^{-1}Ny_N^*(s) \geq 0$ . This is precisely the nonnegativity constraint on  $y_B$  that is relaxed to obtain the Gomory relaxation  $v_B(s)$ . We conclude that the optimal solution  $y_N^*(s)$  of  $v_B(s)$  is feasible, and thus also optimal, for  $v(s)$ .  $\square$

Combining Lemmas 2.3 and 2.8 we observe that the optimal solution of the mixed-integer value function is  $B$ -periodic on  $\Lambda(\bar{d})$ . This  $B$ -periodicity is only valid for sufficiently large right-hand side vectors  $s$ , since it holds on  $\Lambda(\bar{d})$  but not necessarily

on  $\Lambda$ . Moreover, the value of  $\bar{d} := |\det(B)| \sum_{j=1}^n \|N_j\|_2$  may be large depending on the matrix of coefficients  $W$ . For this reason we use the terminology *asymptotic periodicity* to refer to this type of periodicity.

### 2.3. Asymptotic periodicity in mixed-integer programming problems.

Since the results obtained so far hold for every dual feasible basis matrix  $B^k$ ,  $k = 1, \dots, K$ , of the LP-relaxation of  $v$ , we are able to derive a complete characterization of the periodicity properties of  $v$  in Theorem 2.9, the main result of this section.

**THEOREM 2.9.** *Consider the mixed-integer linear programming problem*

$$v(s) = \min\{qy : Wy = s, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}\}, \quad s \in \mathbb{R}^m,$$

where  $W$  is an integer matrix, and  $v(s)$  is finite for all  $s \in \mathbb{R}^m$  by assumptions (A1) and (A2). Then, there exist dual feasible basis matrices  $B^k$  of  $v_{LP}$ ,  $k = 1, \dots, K$ , closed convex polyhedral cones  $\Lambda^k := \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq 0\}$ , distances  $d_k := |\det(B^k)| \sum_{j=1}^n \|N_j^k\|_2$ , and  $B^k$ -periodic functions  $\pi^k$  and  $\psi^k$  such that we have the following:

- (i)  $\bigcup_{k=1}^K \Lambda^k = \mathbb{R}^m$ .
- (ii)  $(\text{int } \Lambda^k) \cap (\text{int } \Lambda^l) = \emptyset$  for every  $k, l \in \{1, \dots, K\}$  with  $k \neq l$ .
- (iii) For every  $s \in \Lambda^k(d_k)$ ,

$$\begin{aligned} y_{B^k}(s) &= (B^k)^{-1}(s - N^k \pi^k(s)), \\ y_{N^k}(s) &= \pi^k(s) \end{aligned}$$

is optimal for  $v(s)$ .

- (iv) For every  $s \in \Lambda^k(d_k)$ ,

$$v(s) = v_{LP}(s) + \psi^k(s),$$

where  $v_{LP}(s)$  is the LP-relaxation of  $v(s)$ , and  $\psi^k = \psi^l$  if  $q_{B^k}(B^k)^{-1} = q_{B^l}(B^l)^{-1}$ .

*Proof.* Consider the LP-relaxation of  $v$ . By the basis decomposition theorem in Walkup and Wets [25], there exist dual feasible basis matrices  $B^k$ ,  $k = 1, \dots, K$ , and corresponding simplicial cones  $\Lambda^k := \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq 0\}$  such that (i) and (ii) hold.

To prove (iii) we let  $k = 1, \dots, K$  be given and we consider the basis matrix  $B^k$ . From Lemma 2.8 we conclude that for every  $s \in \Lambda^k(d_k)$  an optimal solution of  $v(s)$  is given by

$$\begin{aligned} y_{B^k}(s) &:= (B^k)^{-1}(s - N^k y_{N^k}^*(s)), \\ y_{N^k}(s) &:= y_{N^k}^*(s), \end{aligned}$$

where  $y_{N^k}^*(s)$  denotes the optimal solution of the Gomory relaxation  $v_{B^k}(s)$ . The result in (iii) now follows from defining  $\pi^k(s) := y_{N^k}^*(s)$  and observing that  $y_{N^k}^*(s)$  is  $B^k$ -periodic by Lemma 2.3 (iii).

Obviously, if we define  $\psi^k(s) := \bar{q}_{N^k} \pi^k(s) = \bar{q}_{N^k} y_{N^k}^*(s)$ , then  $v(s) = v_{LP}(s) + \psi^k(s)$  for every  $s \in \Lambda^k(d_k)$  and  $\psi^k$  is  $B^k$ -periodic. It remains to show that  $\psi^k = \psi^l$  if  $q_{B^k}(B^k)^{-1} = q_{B^l}(B^l)^{-1}$ . We do so by proving that in this case the Gomory relaxations  $v_{B^k}(s)$  and  $v_{B^l}(s)$  have the same optimal objective value, and thus for every  $s \in \mathbb{R}^m$ ,

$$v_{B^k}(s) = \bar{q}_{N^k} y_{N^k}^*(s) + q_{B^k}(B^k)^{-1}s = \bar{q}_{N^l} y_{N^l}^*(s) + q_{B^l}(B^l)^{-1}s = v_{B^l}(s),$$



implying that  $\psi^k(s) = \bar{q}_{N^k} y_{N^k}^*(s) = \bar{q}_{N^l} y_{N^l}^*(s) = \psi^l(s)$  since  $q_{B^k}(B^k)^{-1} = q_{B^l}(B^l)^{-1}$ .

The key observation in the proof of  $v_{B^k}(s) = v_{B^l}(s)$  is that if  $(\bar{q}_N)_j = 0$ , then relaxing the nonnegativity constraint on the  $j$ th component of  $y_N$  in the Gomory relaxation  $v_B(s)$  does not change its optimal objective value, since we can argue that without loss of generality  $0 \leq (y_N^*(s))_j \leq |\det(B)|$ , similarly to the proof of Lemma 2.3. Since the matrices  $B^k$  and  $B^l$  correspond to the same degenerate dual vertex when  $q_{B^k}(B^k)^{-1} = q_{B^l}(B^l)^{-1}$ , it follows that the columns of  $B^l$  not in  $B^k$  correspond to zero components of  $\bar{q}_{N^k}$  and vice versa. This implies that the optimization problem obtained by relaxing all nonnegativity constraints in  $v_{B^k}$  of nonbasic variables  $y_{N^k}$  corresponding to columns of  $B^l$  not in  $B^k$  is the same as that obtained by relaxing all nonnegativity constraints in  $v_{B^l}$  of nonbasic variables  $y_{N^l}$  corresponding to columns of  $B^k$  not in  $B^l$ , and has the same objective value as both  $v_{B^k}(s)$  and  $v_{B^l}(s)$ . Thus,  $v_{B^k}(s) = v_{B^l}(s)$  for every  $s \in \mathbb{R}^m$ .  $\square$

**3. Convex approximation of the recourse function.** In this section we construct a convex approximation  $\hat{Q}$  of the mixed-integer recourse function  $Q$  using the results from Theorem 2.9. The main idea is to approximate the  $B^k$ -periodic functions  $\psi^k$  by constants  $\Gamma_k$ . In this way the approximating value function  $\hat{v}$  is convex, and thus the recourse approximation  $\hat{Q}(z) := \mathbb{E}_\omega[\hat{v}(\omega - z)]$  is convex. This approach contrasts strongly with the mainstream literature, where the primary approach is to use mixed-integer programming based methods to obtain (near)-optimal solutions.

In section 3.1 we define  $\hat{v}$  and  $\hat{Q}$ , and in section 3.2 we derive properties of the approximating value function  $\hat{v}$  that will be used to bound  $\|Q - \hat{Q}\|_\infty$  in section 5.

**3.1. The approximating second-stage value function  $\hat{v}$ .** By Theorem 2.9, on  $\Lambda^k(d_k)$  the mixed-integer value function  $v$  is the sum of a linear and a periodic function:

$$v(s) = q_{B^k}(B^k)^{-1}s + \psi^k(s), \quad s \in \Lambda^k(d_k),$$

where  $\psi^k$  is  $B^k$ -periodic. To obtain a convex approximation  $\hat{v}$  of  $v$  we replace  $\psi^k$  by a constant  $\Gamma_k$  defined as

$$\Gamma_k := p_k^{-m} \int_0^{p_k} \cdots \int_0^{p_k} \psi^k(x) dx_1 \cdots dx_m,$$

with  $p_k := |\det(B^k)|$ . This constant  $\Gamma_k$  can be interpreted as the “average” of  $\psi^k$ . Obviously, replacing  $\psi^k$  by any other convex function  $\hat{\psi}^k$  also yields a convex approximation  $\hat{v}$  of  $v$ . However,  $\hat{\psi}^k \equiv \Gamma_k$  is the only convex approximation  $\hat{\psi}^k$  of  $\psi^k$  leaving  $\|\psi^k - \hat{\psi}^k\|_\infty$  finite and satisfying  $\int_0^{p_k} \cdots \int_0^{p_k} (\psi^k(x) - \hat{\psi}^k(x)) dx_1 \cdots dx_m = 0$ . The latter will be crucial in our subsequent analysis.

*Remark 3.1.* Notice that  $\Gamma_k \geq 0$  for every  $k = 1, \dots, K$ , since by Theorem 2.9  $\psi^k(s) = \bar{q}_{N^k} y_{N^k}^*(s) \geq 0$  for all  $s \in \mathbb{R}^m$ . Here, we use that the reduced costs satisfy  $\bar{q}_{N^k} \geq 0$  and that the optimal solutions  $y_{N^k}^*(s)$  of the Gomory relaxation  $v_{B^k}(s)$  are nonnegative.

We define  $\hat{v}$  as the pointwise maximum of the affine functions  $q_{B^k}(B^k)^{-1}s + \Gamma_k$  so that  $\hat{v}$  is indeed convex on  $\mathbb{R}^m$ .

**DEFINITION 3.2.** Consider the mixed-integer value function  $v$  as defined in (1.2) and let  $B^k$  and  $\psi^k$  denote the basis matrices and  $B^k$ -periodic functions, respectively, of Theorem 2.9. Then, we define the approximating value function  $\hat{v}$  of  $v$  as

$$\hat{v}(s) := \max_{k=1, \dots, K} \{q_{B^k}(B^k)^{-1}s + \Gamma_k\}, \quad s \in \mathbb{R}^m,$$

where  $\Gamma_k := p_k^{-m} \int_0^{p_k} \cdots \int_0^{p_k} \psi^k(x) dx_1 \cdots dx_m$  with  $p_k := |\det(B^k)|$ .

*Example 3.3.* Consider the simple mixed-integer second-stage value function

$$v(s) = \min\{y_1 + 2y_2 + 2y_3 : y_1 + y_2 - y_3 = s, y_1 \in \mathbb{Z}_+, y_2, y_3 \in \mathbb{R}_+\}, \quad s \in \mathbb{R},$$

with LP-relaxation  $v_{LP}$  equal to

$$v_{LP}(s) = \max\{s, -2s\} = \begin{cases} s & \text{if } s \geq 0, \\ -2s & \text{if } s \leq 0. \end{cases}$$

As can be observed from the expression of  $v_{LP}$ , there are two dual feasible basis matrices  $B^1 = [1]$  and  $B^2 = [-1]$ , and closed convex polyhedral cones  $\Lambda^1 = \mathbb{R}_+$  and  $\Lambda^2 = \mathbb{R}_-$ . The first pair corresponds to the decision variable  $y_1$  and the second to  $y_3$ .

For this particular example it is possible to obtain a closed-form expression of  $v$  so that, using the notation of Theorem 2.9,  $v(s) = v_{LP}(s) + \psi^k(s)$  if  $s \in \Lambda^k$ ,  $k = 1, 2$ , with

$$\psi^1(s) = \begin{cases} s - \lfloor s \rfloor & \text{if } s - \lfloor s \rfloor \leq 3/4, \\ 3 - 3(s - \lfloor s \rfloor) & \text{if } s - \lfloor s \rfloor \geq 3/4, \end{cases}$$

and  $\psi^2(s) = 0$  for every  $s \in \mathbb{R}$ . Thus,  $\Gamma_1 = \int_0^1 \psi^1(s) ds = 3/8$  and  $\Gamma_2 = 0$  so that the approximating value function  $\hat{v}$  defined in Definition 3.2 is given by

$$\hat{v}(s) = \max\{s + 3/8, -2s\}, \quad s \in \mathbb{R}.$$

In Figure 1 we have depicted the second-stage value function  $v$  and its approximating value function  $\hat{v}$ . Observe that  $\hat{v}$  is convex whereas  $v$  is not.

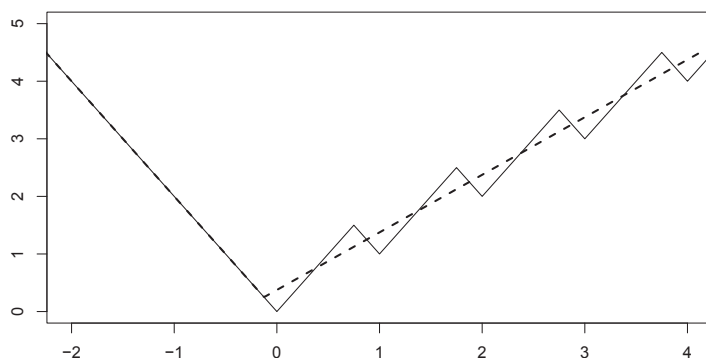


FIG. 1. The simple mixed-integer second stage value function  $v$  of Example 3.3 (solid) and its convex approximating value function  $\hat{v}$  (dashed).

The approximating value function  $\hat{v}$  can be interpreted as a shifted version of the LP-relaxation of  $v$ . The value function  $v_{LP}$  of this LP-relaxation equals  $v_{LP}(s) = \max_{k=1,\dots,K} q_{B^k} (B^k)^{-1}s$  for every  $s \in \mathbb{R}^m$ , and thus  $\hat{v}$  is obtained by shifting every

linear function  $q_{B^k}(B^k)^{-1}s$  by  $\Gamma_k$ . In fact, for the special case of totally unimodular (TU) integer recourse models we obtain  $\hat{v}(s) = v_{LP}(s + \frac{1}{2}e_m)$  with  $e_m$  denoting the  $m$ -dimensional all-one vector.

*Example 3.4.* Let  $W$  be a TU recourse matrix of the form  $W = [W' - I_m]$  with  $W' \in \{0, 1, -1\}^{m \times n_2}$ , and let  $q = [q' \ 0]$  so that the continuous decision variables represent slack variables, and the mixed-integer second-stage value function  $v$  of (1.2) reduces to

$$v(s) = \min\{q'y : W'y \geq s, y \in \mathbb{Z}_+^{n_2}\}, \quad s \in \mathbb{R}^m.$$

Since  $W'$  is an integer matrix we can round up the right-hand side  $s$  to  $\lceil s \rceil$ . Next, we are allowed to relax the integrality constraints since  $W'$  is TU, and we obtain

$$(3.1) \quad v(s) = \min\{q'y : W'y \geq \lceil s \rceil, y \in \mathbb{R}_+^{n_2}\} = v_{LP}(\lceil s \rceil), \quad s \in \mathbb{R}^m.$$

According to Theorem 2.9 there exists dual feasible basis matrices  $B^k$  of  $v_{LP}$ , closed convex cones  $\Lambda^k$ , and distances  $d_k > 0$  such that for every  $s \in \Lambda^k(d_k)$  we have  $v(s) = v_{LP}(s) + \psi^k(s)$ . The expression in (3.1) implies that for every  $k = 1, \dots, K$ ,

$$\psi^k(s) = q_{B^k}(B^k)^{-1}(\lceil s \rceil - s), \quad s \in \mathbb{R}^m,$$

which is indeed  $B^k$ -periodic. Moreover,  $p_k = |\det(B^k)| = 1$  since  $B^k$  is a nonsingular submatrix of the TU matrix  $W$ , and we can obtain the averages  $\Gamma_k$  of  $\psi^k$  by straightforward computation:

$$\Gamma_k = \int_0^1 \cdots \int_0^1 q_{B^k}(B^k)^{-1}(\lceil s \rceil - s) ds_1 \cdots ds_m = \frac{1}{2} q_{B^k}(B^k)^{-1} e_m.$$

We conclude that for every  $s \in \mathbb{R}^m$ ,

$$\begin{aligned} \hat{v}(s) &= \max_{k=1, \dots, K} \left\{ q_{B^k}(B^k)^{-1}s + \Gamma_k \right\} \\ &= \max_{k=1, \dots, K} \left\{ q_{B^k}(B^k)^{-1} \left( s + \frac{1}{2}e_m \right) \right\} \\ &= v_{LP} \left( s + \frac{1}{2}e_m \right). \end{aligned}$$

This is precisely the convex approximation developed by Romeijnnders, van der Vlerk, and Klein Haneveld [17] for this special case. They derive an error bound for this approximation using, among others, the relation between  $v$  and  $v_{LP}$  in (3.1). Moreover, they show that this particular convex approximation has the best bound possible in a worst-case sense.

The convex approximation  $\hat{Q}$  is defined analogously as  $Q(z) := \mathbb{E}_\omega[v(\omega - z)]$ ,  $z \in \mathbb{R}^m$ .

**DEFINITION 3.5.** We define the convex approximation  $\hat{Q}$  of the mixed-integer recourse function  $Q$  defined in (1.1) as

$$\hat{Q}(z) := \mathbb{E}_\omega[\hat{v}(\omega - z)], \quad z \in \mathbb{R}^m,$$

where  $\hat{v}$  is the approximating value function of Definition 3.2.

**3.2. Properties of the approximating value function  $\hat{v}$ .** In this subsection we discuss several properties of the approximating value function  $\hat{v}$ . We will first show that  $\|v - \hat{v}\|_\infty$  is finite, and in the remainder we give a partial characterization of  $\hat{v}$ . That is, we identify areas of the domain of  $\hat{v}$  on which both  $v$  and  $\hat{v}$  attain their value through the same dual feasible basis, i.e., on which both  $\hat{v}(s) = q_{B^k}(B^k)^{-1}s + \Gamma_k$  and  $v(s) = q_{B^k}(B^k)^{-1}s + \psi^k(s)$  hold for the same  $k \in \{1, \dots, K\}$ , and we show that the remainder of the domain can be covered by finitely many hyperslices, to be defined in Definition 3.8.

**LEMMA 3.6.** *Consider the mixed-integer value function  $v$  as defined in (1.2), and its approximating value function  $\hat{v}$  given in Definition 3.2. There exists a constant  $\beta > 0$  such that*

$$\|v - \hat{v}\|_\infty := \sup_{s \in \mathbb{R}^m} |v(s) - \hat{v}(s)| \leq \beta.$$

*Proof.* Let  $v_{LP}$  denote the LP-relaxation of  $v$ . Then, by, e.g., [2] and [4], there exists a constant  $\beta'$  such that

$$\|v - v_{LP}\|_\infty \leq \beta'.$$

Moreover, comparing  $\hat{v}$  and  $v_{LP}$  we observe that  $\|\hat{v} - v_{LP}\|_\infty \leq \max_{k=1, \dots, K} \Gamma_k$  since  $\Gamma_k \geq 0$  for every  $k = 1, \dots, K$  by Remark 3.1. Thus, defining  $\beta := \beta' + \max_{k=1, \dots, K} \Gamma_k$  we have

$$\|v - \hat{v}\|_\infty \leq \|v - v_{LP}\|_\infty + \|v_{LP} - \hat{v}\|_\infty \leq \beta' + \max_{k=1, \dots, K} \Gamma_k = \beta. \quad \square$$

In Proposition 3.7 we show for which values of  $s$  both  $\hat{v}(s) = q_{B^k}(B^k)^{-1}s + \Gamma_k$  and  $v(s) = q_{B^k}(B^k)^{-1}s + \psi^k(s)$  hold. Obviously, the latter holds if  $s \in \Lambda^k(d_k)$ . However, the former does not necessarily hold on the whole of  $\Lambda^k(d_k)$  since a large constant  $\Gamma_j$  may dominate the maximum defining  $\hat{v}$  on  $\Lambda^k(d_k)$ . We will show, however, that on a subset of  $\Lambda^k(d_k)$  this former equality is true, too.

**PROPOSITION 3.7.** *Consider the mixed-integer value function  $v$  as defined in (1.2), and its approximating value function  $\hat{v}$  given in Definition 3.2. Moreover, let  $B^k$ ,  $\psi^k$ ,  $\Lambda^k$ , and  $d^k$ ,  $k = 1, \dots, K$ , denote the basis matrices,  $B^k$ -periodic functions, closed convex polyhedral cones, and distances, respectively, of Theorem 2.9. Then, for every  $k = 1, \dots, K$ , there exists  $\sigma_k \in \Lambda^k(d_k)$  such that for all  $s \in \sigma_k + \Lambda^k \subset \Lambda^k(d_k)$ ,*

$$(3.2) \quad \hat{v}(s) = q_{B^k}(B^k)^{-1}s + \Gamma_k \quad \text{and} \quad v(s) = q_{B^k}(B^k)^{-1}s + \psi^k(s).$$

Moreover, there exists  $b^k \in \mathbb{R}_+^m$  such that  $\sigma_k + \Lambda^k = \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq b^k\}$ .

*Proof.* Let  $k \in \{1, \dots, K\}$  be given. We will show that there exists  $\sigma_k \in \Lambda^k(d_k)$  such that for every  $j \neq k$  and  $s \in \sigma_k + \Lambda^k$ ,

$$(3.3) \quad q_{B^k}(B^k)^{-1}s + \Gamma_k \geq q_{B^j}(B^j)^{-1}s + \Gamma_j.$$

This proves the first equality in (3.2); the second follows immediately from Theorem 2.9 since  $\sigma_k + \Lambda^k \subset \Lambda^k(d_k)$  by Example 2.7.

To prove (3.3), let  $k, j \in \{1, \dots, K\}$  with  $j \neq k$  be given. Since  $B^k$  is an optimal basis matrix of the LP-relaxation of  $v(s)$  for  $s \in \Lambda^k$ , it follows that

$$(3.4) \quad q_{B^k}(B^k)^{-1}s \geq q_{B^j}(B^j)^{-1}s, \quad s \in \Lambda^k.$$

If  $q_{B^k}(B^k)^{-1} = q_{B^j}(B^j)^{-1}$ , then  $\psi^k = \psi^j$  by Theorem 2.9, and thus their averages  $\Gamma_k$  and  $\Gamma_j$  are equal so that (3.3) holds for every  $s \in \Lambda^k$ . Also, if  $\Gamma_k \geq \Gamma_j$ , then (3.3) holds for every  $s \in \Lambda^k$ , so we assume that  $q_{B^k}(B^k)^{-1} \neq q_{B^j}(B^j)^{-1}$  and  $\Gamma_k < \Gamma_j$ . Observe that in this case (3.4) holds with strict inequality for every  $s' \in \text{int } \Lambda^k$ . Thus, for such an  $s' \in \text{int } \Lambda^k$  and  $\alpha > 0$  sufficiently large, we have  $\alpha s' \in \Lambda^k(d_k)$  and

$$\alpha(q_{B^k}(B^k)^{-1}s' - q_{B^j}(B^j)^{-1}s') \geq \Gamma_j - \Gamma_k,$$

so that (3.3) is true with  $s$  replaced by  $\alpha s'$ . Using (3.4) it follows immediately that (3.3) holds for all  $s \in \sigma_{jk} + \Lambda^k$  with  $\sigma_{jk} := \alpha s'$ .

We conclude that (3.3) holds for all  $j \neq k$  and  $s \in \bigcap_{j \neq k} (\sigma_{jk} + \Lambda^k)$ . Moreover, since

$$\sigma_{jk} + \Lambda^k = \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq b^{jk}\}$$

with  $b^{jk} := (B^k)^{-1}\sigma_{jk}$ , it follows that for  $b^k$  defined as the componentwise maximum of  $b^{jk}$ ,  $j \neq k$ , and  $\sigma^k := B^k b^k$ , we have

$$\bigcap_{j \neq k} (\sigma_{jk} + \Lambda^k) = \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq b^k\} = \sigma_k + \Lambda^k. \quad \square$$

Proposition 3.7 shows that on every closed convex polyhedral cone  $\sigma_k + \Lambda^k$ ,  $k = 1, \dots, K$ , we have  $v(s) - \hat{v}(s) = \psi^k(s) - \Gamma_k$ , and thus the difference  $v - \hat{v}$  is  $B^k$ -periodic on  $\sigma_k + \Lambda^k$ . For  $s \in \mathcal{N} := \mathbb{R}^m \setminus (\bigcup_{k=1}^K (\sigma_k + \Lambda^k))$  we do not derive any property of  $v(s) - \hat{v}(s)$  other than the uniform bound of Lemma 3.6. However, we do show in Lemma 3.9 that  $\mathcal{N}$  can be covered by finitely many so-called hyperslices  $H$  of the form  $H := \{x \in \mathbb{R}^m : 0 \leq a^T x \leq \delta\}$ .

**DEFINITION 3.8.** Consider the mixed-integer value function  $v$  as defined in (1.2), and let  $B^k$ ,  $k = 1, \dots, K$ , denote the dual feasible basis matrices of Theorem 2.9 and  $b^k$ ,  $k = 1, \dots, K$ , the translation vectors of Proposition 3.7. Then, for every  $k = 1, \dots, K$  and  $j = 1, \dots, m$ , let  $a_{jk}$  denote the  $j$ th row of  $(B^k)^{-1}$  and  $\delta_{jk}$  the  $j$ th component of  $b^k$ . We define the hyperslice  $H_{jk}$  as

$$H_{jk} := \{t \in \mathbb{R}^m : 0 \leq a_{jk}t \leq \delta_{jk}\}.$$

**LEMMA 3.9.** Consider the hyperslices  $H_{jk}$  from Definition 3.8, and let  $\sigma_k$  and  $\Lambda^k$ ,  $k = 1, \dots, K$ , be defined as in Proposition 3.7. Then,

$$(3.5) \quad \mathcal{N} := \mathbb{R}^m \setminus (\bigcup_{k=1}^K (\sigma_k + \Lambda^k)) \subset \bigcup_{k=1}^K \bigcup_{j=1}^m H_{jk}.$$

*Proof.* Since  $\Lambda^k := \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq 0\}$  and  $\sigma_k + \Lambda^k := \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq b^k\}$ , it follows immediately from the definition of the hyperslices  $H_{jk}$  that

$$\Lambda^k \setminus (\sigma_k + \Lambda^k) \subset \bigcup_{j=1}^m H_{jk}, \quad k = 1, \dots, K.$$

Taking the union over  $k = 1, \dots, K$ , and using  $\bigcup_{k=1}^K \Lambda^k = \mathbb{R}^m$  and  $(\text{int } \Lambda^k) \cap (\text{int } \Lambda^l) = \emptyset$  by Theorem 2.9 (i) and (ii), we obtain the inclusion in (3.5).  $\square$

**4. Total variation bounds.** In this section we develop the tools to bound  $\|Q - \hat{Q}\|_\infty$  in section 5. We will make extensive use of the concept of total variation  $|\Delta|f$  of a one-dimensional probability density function  $f$ .

DEFINITION 4.1. Let  $f : \mathbb{R} \mapsto \mathbb{R}$  be a real-valued function, and let  $I \subset \mathbb{R}$  be an interval. Let  $\Pi(I)$  denote the set of all finite ordered sets  $P = \{x_1, \dots, x_{N+1}\}$  with  $x_1 < \dots < x_{N+1}$  in  $I$ . Then, the total variation of  $f$  on  $I$ , denoted  $|\Delta|f(I)$ , is defined as

$$|\Delta|f(I) = \sup_{P \in \Pi(I)} V_f(P),$$

where

$$V_f(P) = \sum_{i=1}^N |f(x_{i+1}) - f(x_i)|.$$

We will write  $|\Delta|f := |\Delta|f(\mathbb{R})$ .

In section 4.1 we derive a total variation bound for the probability  $\mathbb{P}\{0 \leq a^T \omega \leq \delta\}$  where  $\delta > 0$ ,  $a \in \mathbb{R}^m \setminus \{0\}$ , and  $\omega$  is a continuous random vector, and in section 4.2 we derive a total variation bound for the expectation  $\mathbb{E}_\omega[\mathbb{1}_C(\omega)(\psi(\omega) - \nu)]$ , where  $C \subset \mathbb{R}^m$  is a convex set,  $\mathbb{1}_C(\omega)$  is an indicator function equal to one if  $\omega \in C$ ,  $\psi$  is a  $B$ -periodic function with  $\nu$  representing its average, and  $\omega$  is a continuous random vector with joint pdf  $f$ . Both results will be used in section 5 to derive an error bound for  $\|Q - \hat{Q}\|_\infty$ , the former to bound  $\mathbb{P}\{\omega \in H_{jk}\}$  with  $H_{jk}$  the hyperslice from Definition 3.8, and the latter with  $C = \sigma_k + \Lambda^k$ ,  $\psi = \psi^k$ , and  $\nu = \Gamma_k$ . In both sections 4.1 and 4.2 we assume that all one-dimensional conditional densities of  $f$  are of bounded variation. We let  $\mathcal{H}^m$  denote the set of such joint density functions.

DEFINITION 4.2. A function  $f : \mathbb{R} \mapsto \mathbb{R}$  is of bounded variation if and only if  $|\Delta|f < +\infty$ .

We let  $\mathcal{F}$  denote the set of one-dimensional probability density functions  $f$  of bounded variation.

Remark 4.3. Obviously, by changing a pdf  $f$  on a set of measure zero, the probability distribution does not change. However, the total variation of  $f$  is sensitive to such changes. That is why we assume pdf  $f$  should be left-continuous so that they are well-behaved.

DEFINITION 4.4. For every  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$ , define the  $i$ th conditional density function  $f_i(\cdot|x_{-i})$  of the  $m$ -dimensional joint pdf  $f$  as

$$f_i(x_i|x_{-i}) = \frac{f(x)}{f_{-i}(x_{-i})}, \quad x \in \mathbb{R}^m,$$

with  $x_{-i} \in \mathbb{R}^{m-1}$  representing  $x$  without its  $i$ th component.

We let  $\mathcal{H}^m$  denote the set of all  $m$ -dimensional joint pdf  $f$  whose conditional density functions  $f_i(\cdot|x_{-i})$  are of bounded variation. That is,  $f_i(\cdot|x_{-i}) \in \mathcal{F}$  for all  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$ .

Remark 4.5. For simplicity of exposition, we assume that  $f(x_i|x_{-i})$  is well-defined for all  $x \in \mathbb{R}^m$ . Adjustments for generalizations are obvious but cumbersome.

**4.1. Probability bound.** In this subsection we derive a total variation bound on the probability  $\mathbb{P}\{0 \leq a^T \omega \leq \delta\}$ . The bound shows that for fixed  $a \in \mathbb{R}^m \setminus \{0\}$  and



$\delta > 0$ , this probability converges to 0 if all total variations of the conditional densities of the joint pdf  $f$  of the continuous random vector  $\omega$  converge to 0.

**THEOREM 4.6.** *Let  $\delta > 0$  and  $a \in \mathbb{R}^m \setminus \{0\}$  be given. Then, there exists  $D > 0$  such that for every continuous random vector  $\omega$  with joint pdf  $f \in \mathcal{H}^m$ ,*

$$\mathbb{P}\{0 \leq a^T \omega \leq \delta\} \leq D \sum_{i=1}^m \mathbb{E}_{\omega_{-i}} [|\Delta|f_i(\cdot|\omega_{-i})|].$$

*Proof.* Define  $H := \{x \in \mathbb{R}^m : 0 \leq a^T x \leq \delta\}$  so that  $\mathbb{P}\{0 \leq a^T \omega \leq \delta\} = \int_H f(x) dx$ . Since  $a \neq 0$ , there exists  $j = 1, \dots, m$  such that  $a_j \neq 0$ . By conditioning on all components of  $\omega$  except for the  $j$ th, we have

$$\mathbb{P}\{0 \leq a^T \omega \leq \delta\} = \int_{\mathbb{R}^{m-1}} \int_{H_j(x_{-j})} f_j(x_j|x_{-j}) dx_j f_{-j}(x_{-j}) dx_{-j},$$

where  $H_j(x_{-j}) := \{x_j \in \mathbb{R} : x \in H\}$  equals

$$H_j(x_{-j}) = \left\{ x_j \in \mathbb{R} : -\frac{a_{-j}^T x_{-j}}{a_j} \leq x_j \leq \frac{\delta}{a_j} - \frac{a_{-j}^T x_{-j}}{a_j} \right\} \quad \text{if } a_j > 0,$$

and

$$H_j(x_{-j}) = \left\{ x_j \in \mathbb{R} : \frac{\delta}{a_j} - \frac{a_{-j}^T x_{-j}}{a_j} \leq x_j \leq -\frac{a_{-j}^T x_{-j}}{a_j} \right\} \quad \text{if } a_j < 0.$$

Observe that for every  $x_{-j} \in \mathbb{R}^{m-1}$  and  $a_j \neq 0$ , the interval length  $|H_j(x_{-j})| = |a_j|^{-1} \delta$ . Thus, since  $f_j(x_j|x_{-j}) \leq \frac{1}{2} |\Delta|f_j(\cdot|x_{-j})|$  for every  $x \in \mathbb{R}^m$ , it follows that

$$\begin{aligned} \mathbb{P}\{0 \leq a^T \omega \leq \delta\} &\leq \int_{\mathbb{R}^{m-1}} \frac{1}{2} |a_j|^{-1} \delta |\Delta|f_j(\cdot|x_{-j})| f_{-j}(x_{-j}) dx_{-j} \\ (4.1) \quad &= \frac{1}{2} |a_j|^{-1} \delta \mathbb{E}_{\omega_{-j}} [|\Delta|f_j(\cdot|\omega_{-j})|]. \end{aligned}$$

Defining  $D := \frac{1}{2} |a_j|^{-1} \delta > 0$  and observing that  $\mathbb{E}_{\omega_{-i}} [|\Delta|f_i(\cdot|\omega_{-i})|] \geq 0$  for  $i = 1, \dots, m$ , we obtain

$$\mathbb{P}\{0 \leq a^T \omega \leq \delta\} \leq D \sum_{i=1}^m \mathbb{E}_{\omega_{-i}} [|\Delta|f_i(\cdot|\omega_{-i})|]. \quad \square$$

**Remark 4.7.** Observe that the bound in Theorem 4.6 can be improved by minimizing the expression in (4.1) over  $j = 1, \dots, m$ . However, we prefer to present the result in this way for notational convenience, since the error bound in Theorem 5.1 will also contain terms of the form  $\sum_{i=1}^m \mathbb{E}_{\omega_{-i}} [|\Delta|f_i(\cdot|\omega_{-i})|]$  due to Theorem 4.13 in the next subsection.

**4.2. Bounds on the expectation of  $B$ -periodic functions.** In this subsection we derive total variation error bounds on the expectation of  $B$ -periodic functions  $\psi$ . In fact, we will bound  $\int_{\Lambda} (\psi(x) - \nu) f(x) dx$ , where  $\Lambda \subset \mathbb{R}^m$  is a convex set,  $f \in \mathcal{H}^m$ , and  $\nu$  represents the average of  $\psi$ . To do so we first introduce some auxiliary lemmas on properties of  $B$ -periodic functions  $\psi$  and on extensions of the total variation

bounds on the expectation of one-dimensional periodic functions of Romeijnders, van der Vlerk, and Klein Haneveld [17].

First, we consider properties of  $B$ -periodic functions  $\psi$ .

LEMMA 4.8. *Let  $\psi : \mathbb{R}^m \mapsto \mathbb{R}$  be a  $B$ -periodic function with  $B \in \mathbb{Z}^{m \times m}$  nonsingular. Then,  $\psi$  is  $pI_m$ -periodic with  $p = |\det(B)|$ .*

*Proof.* Let  $x \in \mathbb{R}^m$  and  $\ell \in \mathbb{Z}^m$  be given. We need to show that  $\psi(x + p\ell) = \psi(x)$ . Since  $\psi(x + p\ell) = \psi(x + BB^{-1}(p\ell))$ , the result follows immediately from the  $B$ -periodicity of  $\psi$  if  $B^{-1}(p\ell) \in \mathbb{Z}^m$ . Using  $B^{-1} = (\det(B))^{-1} \text{adj}(B)$ , we can rewrite  $B^{-1}(p\ell)$  as

$$B^{-1}(p\ell) = pB^{-1}\ell = |\det(B)|(\det(B))^{-1} \text{adj}(B)\ell.$$

Since  $B$  is an integer matrix, it follows that  $\text{adj}(B)$  is an integer matrix, and thus  $B^{-1}(p\ell) \in \mathbb{Z}^m$ . We conclude that  $\psi$  is  $pI_m$ -periodic.  $\square$

This implies that we can restrict our attention to  $pI_m$ -periodic functions  $\psi$ . Such functions are periodic in  $x_i$  with period  $p$  for every given  $x_{-i} \in \mathbb{R}^{m-1}$ , so we can apply the one-dimensional total variation bounds of Romeijnders, van der Vlerk, and Klein Haneveld [17]. We use the following notation.

DEFINITION 4.9. *Let  $\psi : \mathbb{R}^m \mapsto \mathbb{R}$  be  $pI_m$ -periodic. We define for all  $i = 1, \dots, m$*

$$\psi_i(\vec{x}_i) := p^{-i} \int_0^p \cdots \int_0^p \psi(x) dx_1 \cdots dx_i,$$

where  $\vec{x}_i := (x_{i+1}, \dots, x_m)$ . Moreover, we define  $\psi_0(\vec{x}_0) := \psi(x)$  and  $\psi_m(\vec{x}_m) := \psi_m$ .

Example 4.10. Let  $m = 2$ . Then,

$$\psi_1(\vec{x}_1) = \psi_1(x_2) := p^{-1} \int_0^p \psi(x_1, x_2) dx_1$$

and

$$\psi_2(\vec{x}_2) = \psi_2 := p^{-2} \int_0^p \int_0^p \psi(x_1, x_2) dx_1 dx_2 = p^{-1} \int_0^p \psi_1(x_2) dx_2.$$

The functions in Definition 4.9 are useful since they allow us to decompose  $\psi(x) - \nu$  with  $\nu := \psi_m(\vec{x}_m) = p^{-m} \int_0^p \cdots \int_0^p \psi(x) dx_1 \cdots dx_m$  as

$$(4.2) \quad \psi(x) - \nu = \sum_{i=1}^m \left( \psi_{i-1}(\vec{x}_{i-1}) - \psi_i(\vec{x}_i) \right), \quad x \in \mathbb{R}^m,$$

where for a given  $\vec{x}_i$  the function  $\psi_{i-1}$  is periodic in  $x_i$  with mean value  $\psi_i(\vec{x}_i)$ .

LEMMA 4.11. *Let  $\psi : \mathbb{R}^m \mapsto \mathbb{R}$  be  $pI_m$ -periodic. Then, for every  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$ , the function  $\psi_{i-1}(\vec{x}_{i-1}) = \psi_{i-1}(x_i, \vec{x}_i)$  is periodic in  $x_i$ , with period  $p$  and finite mean value  $p^{-1} \int_0^p \psi_{i-1}(\vec{x}_{i-1}) dx_i = \psi_i(\vec{x}_i)$ .*

*Proof.* Let  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$  be given. From the definition of  $pI_m$ -periodicity it follows directly that  $\psi_{i-1}(x_i, \vec{x}_i)$  is periodic in  $x_i$ . Moreover, using Definition 4.9 for  $\psi_{i-1}(\vec{x}_{i-1})$ , its mean value equals

$$p^{-1} \int_0^p \psi_{i-1}(\vec{x}_{i-1}) dx_i = p^{-i} \int_0^p \cdots \int_0^p \psi(x) dx_1 \cdots dx_i = \psi_i(\vec{x}_i). \quad \square$$

Next, we derive an extension of Theorem 5 from Romeijnders, van der Vlerk, and Klein Haneveld [17]. With this reference we show that for every one-dimensional periodic function  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  with period  $p$  and finite mean value  $\nu := p^{-1} \int_0^p \varphi(x) dx$ ,

$$(4.3) \quad \left| \mathbb{E}_f[\varphi(\omega)] - \nu \right| \leq \frac{|\Delta|f}{4} \int_0^p |\varphi(x) - \nu| dx,$$

where  $\omega$  is a random variable with probability density function  $f \in \mathcal{F}$ . Lemma 4.12 generalizes this result and will be used in the proof of Theorem 4.13.

LEMMA 4.12. *Let  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  be a periodic function with period  $p$  and finite mean value  $\nu := p^{-1} \int_0^p \varphi(x) dx$ . Moreover, let  $I \subset \mathbb{R}$  denote an interval and assume that  $|\varphi(x) - \nu| \leq \beta$  for all  $x \in \mathbb{R}$ . Then, for every  $f \in \mathcal{F}$ ,*

$$(4.4) \quad \left| \int_I (\varphi(x) - \nu) f(x) dx \right| \leq \frac{|\Delta|f}{4} \beta p.$$

*Proof.* Define  $K_I := \int_I f(x) dx$ . If  $K_I = 0$ , then the claim holds trivially. Otherwise, if  $K_I > 0$ , then define  $g : \mathbb{R} \mapsto \mathbb{R}$  as

$$g(x) = \begin{cases} K_I^{-1} f(x), & x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that  $g$  is a pdf with  $|\Delta|g \leq K_I^{-1} |\Delta|f$  and

$$(4.5) \quad \left| \int_I (\varphi(x) - \nu) f(x) dx \right| = K_I \left| \int_{\mathbb{R}} (\varphi(x) - \nu) g(x) dx \right| = K_I \left| \mathbb{E}_g[\varphi(\omega)] - \nu \right|.$$

Applying (4.3) to the right-hand side of (4.5) yields

$$\left| \int_I (\varphi(x) - \nu) f(x) dx \right| \leq K_I \frac{|\Delta|g}{4} \int_0^p |\varphi(x) - \nu| dx \leq \frac{|\Delta|f}{4} \int_0^p |\varphi(x) - \nu| dx.$$

The inequality in (4.4) holds since by assumption  $|\varphi(x) - \nu| \leq \beta$  for all  $x \in \mathbb{R}$  and thus  $\int_0^p |\varphi(x) - \nu| dx \leq \beta p$ .  $\square$

Now we are ready to prove the main result of this subsection.

THEOREM 4.13. *Let  $\psi : \mathbb{R}^m \mapsto \mathbb{R}$  be a  $B$ -periodic function with finite mean value  $\nu := p^{-m} \int_0^p \cdots \int_0^p \psi(x) dx_1 \cdots dx_m$ , where  $B \in \mathbb{Z}^{m \times m}$  is nonsingular. Assume that there exists  $\beta > 0$  such that  $|\psi(x) - \nu| \leq \beta$  for all  $x \in \mathbb{R}^m$ . Then, for every convex set  $\Lambda \subset \mathbb{R}^m$  and every continuous random vector  $\omega$  with joint probability density function  $f \in \mathcal{H}^m$ ,*

$$\left| \int_{\Lambda} (\psi(x) - \nu) f(x) dx \right| \leq \frac{1}{2} \beta |\det(B)| \sum_{i=1}^m \mathbb{E}_{\omega_{-i}} \left[ |\Delta|f_i(\cdot | \omega_{-i}) \right].$$

*Proof.* From Lemma 4.8 it follows that  $\psi$  is  $pI_m$ -periodic with  $p = |\det(B)|$ . Thus, using the notation from Definition 4.9 and the expression for  $\psi(x) - \nu$  in (4.2), we have

$$\begin{aligned} \left| \int_{\Lambda} (\varphi(x) - \nu) f(x) dx \right| &= \left| \int_{\Lambda} \sum_{i=1}^m \left( \psi_{i-1}(\vec{x}_{i-1}) - \psi_i(\vec{x}_i) \right) f(x) dx \right| \\ &\leq \sum_{i=1}^m \left| \int_{\Lambda} \left( \psi_{i-1}(\vec{x}_{i-1}) - \psi_i(\vec{x}_i) \right) f(x) dx \right|, \end{aligned}$$

where the inequality holds by interchanging summation and integration, and by applying the triangle inequality. We proceed by conditioning on all components except for the  $i$ th, obtaining

$$\begin{aligned} \left| \int_{\Lambda} (\varphi(x) - \nu) f(x) dx \right| &\leq \sum_{i=1}^m \left| \int_{\mathbb{R}^{m-1}} G_i(x_{-i}) f_{-i}(x_{-i}) dx_{-i} \right| \\ &\leq \sum_{i=1}^m \int_{\mathbb{R}^{m-1}} |G_i(x_{-i})| f_{-i}(x_{-i}) dx_{-i}, \end{aligned}$$

where for every  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$ ,

$$G_i(x_{-i}) := \int_{\Lambda_i(x_{-i})} \left( \psi_{i-1}(\vec{x}_{i-1}) - \psi_i(\vec{x}_i) \right) f_i(x_i | x_{-i}) dx_i,$$

and where  $\Lambda_i(x_{-i}) := \{x_i \in \mathbb{R} : x \in \Lambda\}$  is a convex set. By Lemma 4.11 the function  $\psi_{i-1}(\vec{x}_{i-1})$  is periodic in  $x_i$  with period  $p$  and mean value  $\psi_i(\vec{x}_i)$  for every  $i = 1, \dots, m$  and  $x_{-i} \in \mathbb{R}^{m-1}$ . Moreover,  $|\psi_{i-1}(\vec{x}_{i-1}) - \psi_i(\vec{x}_i)| \leq 2\beta$  for all  $x \in \mathbb{R}^m$  since  $|\psi(x) - \nu| \leq \beta$  for all  $x \in \mathbb{R}^m$ , and thus the  $|G_i(x_{-i})|$  can be bounded using (4.4) in Lemma 4.12. We have

$$\begin{aligned} \left| \int_{\Lambda} (\varphi(x) - \nu) f(x) dx \right| &\leq \sum_{i=1}^m \int_{\mathbb{R}^{m-1}} \frac{1}{2} \beta |\det(B)| |\Delta| f_i(\cdot | x_{-i}) f_{-i}(x_{-i}) dx_{-i} \\ &= \frac{1}{2} \beta |\det(B)| \sum_{i=1}^m \mathbb{E}_{\omega_{-i}} \left[ |\Delta| f_i(\cdot | \omega_{-i}) \right]. \quad \square \end{aligned}$$

**5. Error bound for convex approximation  $\hat{Q}$ .** After the technical preparations of the previous section we are ready to derive an upper bound for  $\|Q - \hat{Q}\|_{\infty}$ , the main result of this paper.

**THEOREM 5.1.** *Consider the mixed-integer recourse function*

$$Q(z) := \mathbb{E}_f \left[ \min \{ qy : Wy = \omega - z, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3} \} \right], \quad z \in \mathbb{R}^m,$$

and let  $\hat{Q}$  denote its convex approximation defined as

$$\hat{Q}(z) := \mathbb{E}_f \left[ \hat{v}(\omega - z) \right], \quad z \in \mathbb{R}^m,$$

where  $\hat{v}$  is the approximating value function from Definition 3.2. Then, there exists a constant  $C > 0$  such that for every continuous random vector  $\omega$  with joint probability density function  $f \in \mathcal{H}^m$ ,

$$\|Q - \hat{Q}\|_{\infty} \leq C \sum_{i=1}^m \mathbb{E}_{\omega_{-i}} \left[ |\Delta| f_i(\cdot | \omega_{-i}) \right].$$

*Proof.* Let  $f \in \mathcal{H}^m$  and  $z \in \mathbb{R}^m$  be given, and define  $\hat{\omega} := \omega - z$ . Then,  $\hat{\omega}$  has pdf  $g \in \mathcal{H}^m$  defined as  $g(x) = f(x + z)$ . We consider  $|Q(z) - \hat{Q}(z)|$  and we rewrite this

difference as

$$\begin{aligned} |Q(z) - \hat{Q}(z)| &= \left| \mathbb{E}_f[v(\omega - z) - \hat{v}(\omega - z)] \right| \\ &= \left| \mathbb{E}_g[v(\hat{\omega}) - \hat{v}(\hat{\omega})] \right| \\ (5.1) \quad &\leq \sum_{k=1}^K \left| \int_{\sigma_k + \Lambda^k} (v(x) - \hat{v}(x))g(x)dx \right| \end{aligned}$$

$$(5.2) \quad + \left| \int_{\mathcal{N}} (v(x) - \hat{v}(x))g(x)dx \right|.$$

Here,  $\Lambda^k := \{t \in \mathbb{R}^m : (B^k)^{-1}t \geq 0\}$  are the closed convex cones from Theorem 2.9, with  $B^k$  denoting the dual feasible basis matrices of the LP-relaxation of  $v$ , and the vectors  $\sigma_k$  are defined in Proposition 3.7 such that  $v(s) - \hat{v}(s) = \psi^k(s) - \Gamma_k$  for  $s \in \sigma_k + \Lambda^k$ . Moreover, by Theorem 2.9 the functions  $\psi^k$  are  $B^k$ -periodic with mean value

$$\Gamma_k := p_k^{-m} \int_0^{p_k} \cdots \int_0^{p_k} \psi^k(x) dx_1 \cdots dx_m$$

with  $p_k := |\det(B^k)|$ . Since by Lemma 3.6 there exists  $\beta > 0$  such that  $|\psi^k(s) - \Gamma_k| \leq \beta$  for every  $s \in \mathbb{R}^m$  and since  $\sigma_k + \Lambda^k$  is convex, all assumptions of Theorem 4.13 hold, and thus for every  $k = 1, \dots, K$ ,

$$(5.3) \quad \left| \int_{\sigma_k + \Lambda^k} (v(x) - \hat{v}(x))g(x)dx \right| \leq \frac{1}{2}\beta |\det(B^k)| \sum_{i=1}^m \mathbb{E}_{\hat{\omega}_{-i}} \left[ |\Delta|g_i(\cdot|\hat{\omega}_{-i}) \right].$$

Next, consider  $\left| \int_{\mathcal{N}} (v(x) - \hat{v}(x))g(x)dx \right|$  with  $\mathcal{N} := \mathbb{R}^m \setminus (\cup_{k=1}^K (\sigma_k + \Lambda^k))$ . Again using  $|v(s) - \hat{v}(s)| \leq \beta$  for all  $s \in \mathbb{R}^m$  by Lemma 3.6 yields

$$\left| \int_{\mathcal{N}} (v(x) - \hat{v}(x))g(x)dx \right| \leq \beta \mathbb{P}\{\hat{\omega} \in \mathcal{N}\}.$$

By Lemma 3.9 it follows that  $\mathcal{N}$  can be covered by the hyperslices  $H_{jk} := \{t \in \mathbb{R}^m : 0 \leq a_{jk}^T t \leq \delta_{jk}\}$  given in Definition 3.8. Using this observation and applying the union bound we obtain

$$\begin{aligned} \left| \int_{\mathcal{N}} (v(x) - \hat{v}(x))g(x)dx \right| &\leq \beta \mathbb{P}\left\{ \hat{\omega} \in \bigcup_{k=1}^K \bigcup_{j=1}^m H_{jk} \right\} \\ &\leq \beta \sum_{k=1}^K \sum_{j=1}^m \mathbb{P}\{\hat{\omega} \in H_{jk}\}. \end{aligned}$$

For every  $k = 1, \dots, K$  and  $j = 1, \dots, m$ , we bound  $\mathbb{P}\{\hat{\omega} \in H_{jk}\}$  using Theorem 4.6. Hence, there exist constants  $D_{jk} > 0$  such that

$$(5.4) \quad \left| \int_{\mathcal{N}} (v(x) - \hat{v}(x))g(x)dx \right| \leq \left( \beta \sum_{k=1}^K \sum_{j=1}^m D_{jk} \right) \sum_{i=1}^m \mathbb{E}_{\hat{\omega}_{-i}} \left[ |\Delta|g_i(\cdot|\hat{\omega}_{-i}) \right].$$

Substituting (5.3) and (5.4) into (5.2) yields

$$|Q(z) - \hat{Q}(z)| \leq \left( \frac{1}{2}\beta \sum_{k=1}^K |\det(B^k)| + \beta \sum_{k=1}^K \sum_{j=1}^m D_{jk} \right) \sum_{i=1}^m \mathbb{E}_{\hat{\omega}_{-i}} \left[ |\Delta|g_i(\cdot|\hat{\omega}_{-i}) \right].$$

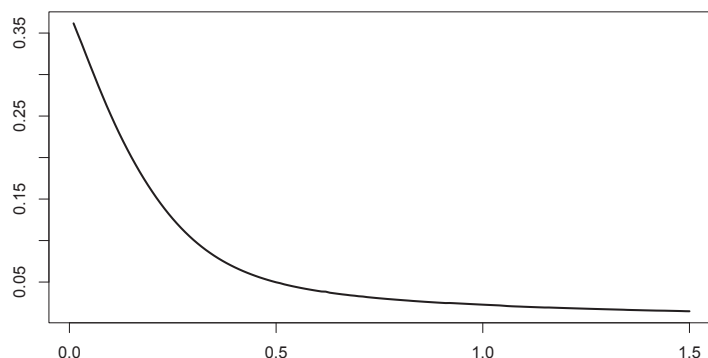


FIG. 2. The supremum norm  $\|Q - \hat{Q}\|_\infty$  for the simple mixed-integer recourse function  $Q$  of Example 5 as a function of  $\sigma$ , the standard deviation of the normally distributed random variable  $\omega$  with mean  $\mu = 0$ .

Finally, we define  $C := \frac{1}{2}\beta \sum_{k=1}^K |\det(B^k)| + \beta \sum_{k=1}^K \sum_{j=1}^m D_{jk} > 0$  and obtain the desired result since  $\mathbb{E}_{\hat{\omega}_{-i}}[|\Delta|g_i(\cdot|\hat{\omega}_{-i})] = \mathbb{E}_{\omega_{-i}}[|\Delta|f_i(\cdot|\omega_{-i})]$  for every  $i = 1, \dots, m$ .  $\square$

*Example 5.2.* Consider the simple mixed-integer second-stage value function  $v$  from Example 3.3, and suppose that  $\omega$  is a normal random variable with mean  $\mu$  and variance  $\sigma^2$ . Then, the pdf  $f$  is unimodal with maximum  $1/\sqrt{2\pi\sigma^2}$  at  $x = \mu$  so that  $|\Delta|f = 2/\sqrt{2\pi\sigma^2} = \sigma^{-1}\sqrt{2/\pi}$ . Thus, according to Theorem 5.1 there exists a constant  $C > 0$  such that  $\|Q - \hat{Q}\|_\infty \leq C\sigma^{-1}\sqrt{2/\pi}$ . In Figure 2 we show the value of  $\|Q - \hat{Q}\|_\infty$ , obtained using brute force computation, as a function of  $\sigma$ ; the mean  $\mu$  equals 0. We observe that  $\|Q - \hat{Q}\|_\infty$  indeed decreases (approximately) hyperbolically in  $\sigma$  as its upper bound  $C\sigma^{-1}\sqrt{2/\pi}$  suggests.

*Example 5.3.* Again consider the simple mixed-integer second-stage value function  $v$  from Example 3.3, but now suppose that  $\omega$  is exponentially distributed with mean  $1/\lambda$ . Then, the pdf  $f$  of  $\omega$  is unimodal with maximum  $\lambda$  at  $x = 0$  so that  $|\Delta|f = 2\lambda$ . Thus, by Theorem 5.1 there exists a constant  $C > 0$  such that  $\|Q - \hat{Q}\|_\infty \leq C\lambda$ . We conclude that the error bound converges to zero if  $\lambda \rightarrow 0$ .

The bound in Theorem 5.1 shows that  $\|Q - \hat{Q}\|_\infty \rightarrow 0$  as all total variations of the density functions of the random variables in the model converge to zero. For example, for normal density functions this is the case if the standard deviations  $\sigma \rightarrow +\infty$ ; see Example 5. In fact, Theorem 5.1 implies that *any* mixed-integer recourse function  $Q$  can be approximated reasonably well by a convex approximation  $\hat{Q}$  if the total variations of the density functions of the random variables in the model are small enough.

**6. Discussion.** We consider two-stage recourse models with randomness in the right-hand side, where the second stage is a mixed-integer linear program. Inspired by and generalizing results of Gomory [7], we derive asymptotic periodicity results for the second-stage mixed-integer value function. Based on these results we construct a new convex approximation  $\hat{v}$  of  $v$  that can be considered as a shifted LP-relaxation. The corresponding convex approximation  $\hat{Q}(z) := \mathbb{E}_\omega[\hat{v}(\omega - z)]$ ,  $z \in \mathbb{R}^m$ , of the recourse



function  $Q$  coincides with that of Romeijnnders, van der Vlerk, and Klein Haneveld [17] for the special case of totally unimodular integer recourse models, in which case it is the convex approximation with the best worst-case error bound possible.

We prove an error bound for  $\hat{Q}$  (by deriving an upper bound on  $\|Q - \hat{Q}\|_\infty$ ) that depends on the total variations of the probability density functions of the random variables in the model, and that converges to zero as these total variations converge to zero. This implies that *any* mixed-integer recourse function  $Q$  can be approximated well by  $\hat{Q}$  if these total variations are small enough, or in other words, if the “variability” of the randomness in the model is large enough.

The results in this paper are the first of this kind in the general setting of mixed-integer recourse models. In fact, they are the first results more general than those for the TU integer recourse case mentioned earlier. As such, it is not surprising that the error bound for the convex approximation  $\hat{Q}$  in Theorem 5.1 is asymptotic in nature: we merely show the existence of a constant  $C > 0$  such that  $\|Q - \hat{Q}\|_\infty \leq C\theta(f)$  for every continuous random vector  $\omega$  with joint pdf  $f \in \mathcal{H}^m$ , where  $\theta(f)$  depends on the total variations of the joint pdf  $f$ .

Although it is possible to obtain a closed-form expression for  $C$  based on the analysis in this paper, we do not present such an expression here, mainly because the value of  $C$  will depend strongly on  $K$ , the number of dual feasible basis matrices of the LP-relaxation  $v_{LP}$  of  $v$ , which generally increases exponentially in the size of the second-stage mixed-integer program. For this reason, the error bound may be too large for practical purposes, even if the actual error is reasonably small, and further research is needed to sharpen the bound.

Such a (practically meaningful) sharper bound might be hard to obtain in full generality, but may first be obtained for special cases or particular problem instances, where the structure at hand can be exploited. For example, for the TU integer recourse case in [17], a dual representation of the second-stage value function  $v$  is used to obtain such an error bound of  $\hat{Q}$  that is much sharper than the one presented here, and in fact, does not depend on the number  $K$  of dual feasible basis matrices of  $v_{LP}$ .

From a computational point of view there are also several issues to be considered. The most important is the computation of the  $K$  constants  $\Gamma_k$ , since there may be a large number of them. This implies that for large problem instances, again some sort of approximation may be needed, or alternatively, for special cases such as the TU integer recourse case in Example 3.4, closed-form expressions for the  $\Gamma_k$  may be obtained, and the approximating value function  $\hat{v}$  may be computationally tractable. In any case, further research into these computational issues is required.

Acknowledging these computational issues, we would like to stress (again) the theoretical contribution of our convex approximation  $\hat{Q}$  and its associated error bound, which also gives insights into the behavior of the mixed-integer recourse function  $Q$ . For example, if the error bound is small, then  $Q$  is approximately convex and might be treated *as though it were* convex. On the other hand, if the error bound is large, then the approximating solution  $\hat{x}$  may be used as an initial solution in some meta-heuristic, or as a feasible solution in a branch-and-bound scheme, speeding up computations in the latter case if the solution  $\hat{x}$  is reasonably good.

Other directions for future research include extending the analysis to multistage mixed-integer recourse models. Alternatively, for the two-stage case random cost parameters  $q(\omega)$  and random technology matrices,  $T(\omega)$  may be considered.

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